Uniform time-decay of semigroups of contraction

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Abstract. We discuss the uniform time-decay of semi-groups generated by dissipative Schrödinger operators in the semiclassical regime.

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High frequency analysis of propagation of waves in media with variable absorption index leads to the following dissipative Schrödinger equation:

$$\begin{cases} \mathrm{i}h\frac{\partial}{\partial t}u^h(x,t) &= P(h)u^h(x,t), \\ u^h(x,0) &= u_0^h(x), \end{cases}$$

where $P(h) = -h^2\Delta + V_1(x) - ihV_2(x)$, $x \in \mathbb{R}^n$, $h \in]0, h_0]$ is a small parameter proportional to wave length and V_j, $j = 1, 2$, are real functions with $V_2 \geq 0$ and $V_2 \neq 0$. Assume that V_j is smooth, satisfying for some $\rho > 0$

$$|\partial_x^\alpha V_j(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad j = 1, 2;$$

here $\langle x\rangle = (1 + |x|^2)^{1/2}$. Let $S_h(t) = e^{-itP(h)/h}$, $t \geq 0$, be the associated semigroup of contractions in $L^2(\mathbb{R}^n)$. Then $\|S_h(t)\| \leq 1$ for all $t \geq 0$ and $h \in]0, h_0]$. The interplay between propagation along the flow of the Hamiltonian $p_1(x, \xi) = \xi^2 + V_1(x)$ and the dissipation governs the long-time behavior of solutions. A natural question in this connection is the following

Question. Can one establish a uniform time-decay estimate for the semigroup $S_h(t)$ in the form

$$\|\langle x\rangle^{-s} S_h(t)\langle x\rangle^{-s}\| \leq w(t), \quad t > 0, \tag{1}$$

uniformly in $h \in]0, h_0]$? Here $s > 0$ and $w(t)$ is independent of h and such that $w(t) \to 0$ as $t \to \infty$.

When $n = 3$ and $\rho > 2$, one can show that *for each fixed* $h > 0$, one can take $w(t) = C_h\langle t\rangle^{-r}$ for any $r < s$ and $r \in [0, 3/2]$. In the regime $h \to 0$, long-time behaviors of the quantum evolution are closely related to the classical dynamics. Let $(x(t; y, \eta), \xi(t; y, \eta))$ denote the classical Hamiltonian flow of

$p_1(x,\xi)$ with initial data (y,η). Making use of Egorov's theorem, one can deduce that a necessary condition for (1) to be true is that

$$|\langle x(t;y,\eta)\rangle^{-s} e^{-2\int_0^t V_2(x(\tau;y,\eta))d\tau}\langle y\rangle^{-s}| \le w(t) \tag{2}$$

for all $(y,\eta)\in\mathbb{R}^{2n}$. Since $w(t)$ tends to zero, estimate (2) implies that each bounded classical trajectory should pass through the open set $\{x; V_2(x)>0\}$. In addition, one derives from (2) that $w(t)\ge C\langle t\rangle^{-s\sigma}$ with $\sigma=\min\{1,\tau_0\}$, where τ_0 is the divergence rate in t of non-trapping trajectories with energy 0. Is the condition (2) sufficient for a uniform time-decay estimate of the form (1)? If yes, can one take $w(t)=C\langle t\rangle^{-\min\{\frac{n}{2},s\sigma\}}$? The restriction on the decay rate by $\frac{n}{2}$ comes from the threshold behavior of the semigroup for fixed h.

Recall that in the selfadjoint case ($V_2=0$) and with a localization in energies away from $\mathbb{R}_-$, the result is true with $w(t)=C_s\langle t\rangle^{-s}$ for any $s>0$. More precisely, let $U(t,h)=e^{-\mathrm{i}tP_1(h)/h}$, $P_1(h)=-h^2\Delta+V_1$, $I=]a,b[$ with $a>0$, $\chi\in C_0^\infty(I)$. Then the estimate

$$\|\langle x\rangle^{-s}\chi(P_1(h))U(t,h)\langle x\rangle^{-s}\|\le C_s\langle t\rangle^{-\epsilon},\quad t\in\mathbb{R}, \tag{3}$$

holds for some $s,\epsilon>0$ and uniformly in $h\in]0,h_0]$ *if and only if* every energy E in supp χ is non-trapping. If the latter is satisfied, (3) holds with $\epsilon=s$ and for any $s>0$; see [2].

For non-selfadjoint operators, one can no longer use compactly supported cut-off. The main difficulty to prove a result like (1) is the semiclassical analysis near the threshold zero for dissipative Schrödinger operators. A closely related problem is a global limiting absorption principle on the whole real axis from the the upper half-complex plane and a nice resolvent estimate in $h>0$. For energies away from zero and under the condition (2), this was recently obtained by J. Royer (see [1]). The question is open near the threshold zero.

References

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